LATTICE STRUCTURES FOR QUANTUM CHANNELS

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Abstract. We suggest that a certain one-to-one parametrization of completely positive maps on the matrix algebra $\mathcal{M}_n$ might be useful in the study of quantum channels. This is illustrated in the case of binary quantum channels. While the algorithm is quite intricate, it admits a simple, lattice structure representation.

1. Introduction

Some recent papers deal with the analysis of the completely positive, trace-preserving linear maps on the matrix algebra $\mathcal{M}_n$, [4], [9]. The analysis is quite complete in the case $n = 2$, as it can be seen in the paper [9]. The purpose of this paper is to introduce an algorithm that tests the complete positivity of a linear map on $\mathcal{M}_n$, for any $n \geq 2$. This appears as a sort of Schur-Cohn test and it allows the introduction of certain lattice structures associated to completely positive linear maps. The algorithm is applied to $\mathcal{M}_2$ and the result is compared with the analysis in [9]. Since our algorithm produces a "free" parametrization of the completely positive maps on $\mathcal{M}_n$, it is nonlinear in nature and other applications in order to check its usefulness remain to be investigated.

2. Completely positive maps on $\mathcal{M}_n$

Let $\mathcal{M}_n$ denote the algebra of complex $n \times n$ matrices. A linear map $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ from a C^*-algebra into the set $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$ is called *completely positive* if for every positive integer n, the map

$$\Phi \otimes I_{\mathcal{M}_n} : \mathcal{A} \otimes \mathcal{M}_n \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_n$$

is positivity preserving. By Stinespring Theorem, [11], [8], any such map is the compression of a $*$-homomorphism. For linear completely positive maps on $\mathcal{M}_n$, this implies a somewhat more explicit representation of the form:

$$\Phi(X) = \sum_j A_j^* X A_j, \tag{2.1}$$

where $\{A_j\}$ is a finite set of elements in $\mathcal{M}_n$, [6], [3], [8]. The representation (2.1) is not unique and another characterization of linear completely positive maps on $\mathcal{M}_n$ is also useful. Thus, by a result of Choi [3], [8], the linear map $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ is completely

positive if and only if the matrix

$$S = S_\Phi = \begin{bmatrix} \Phi(E_{11}) & \dots & \Phi(E_{1n}) \\ \\ \vdots & \ddots & \\ \\ \Phi(E_{n1}) & & \Phi(E_{nn}) \end{bmatrix} \tag{2.2}$$

is positive (semi-definite), where $\{E_{kj}\}_{k,j=1}^n$ are the standard matrix units of $\mathcal{M}_n$, that is, E_{kj} is 1 in the (k,j)-th entry and 0 elsewhere. We notice that if $X = [X_{kj}]_{k,j=1}^n$, then $Y = [Y_{kj}]_{k,j=1}^n = \Phi(X)$ is given by the relations

$$Y_{kj} = \sum_{l,m} \Phi(E_{lm})_{kj} X_{lm}, \tag{2.3}$$

and the correspondence (2.2) between the completely positive maps on $\mathcal{M}_n$ and the $n^2 \times n^2$ positive matrices is one-to-one and affine (see [4] for details).

Of special interest in quantum information are those linear completely positive maps that preserve the trace. Such maps are usually called *quantum channels*, [1]. The adjoint $\hat{\Phi}$ of a linear map Φ on $\mathcal{M}_n$ is defined with respect to the Hilbert structure on $\mathcal{M}_n$ given by the Hilbert-Schmidt inner product (linear in the first variable), $\langle A, B\rangle = TrAB^*$, $A, B \in \mathcal{M}_n$, where B^* denotes the usual adjoint of the operator B. It follows that Φ is trace-preserving if and only if $\hat{\Phi}$ is unital ($\hat{\Phi}(I) = I$).

We will use some standard notation associated to contractions on Hilbert spaces. Thus, let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all bounded linear maps operators from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$. The operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *contraction* if $\|T\| \leq 1$. The *defect operator* of T is $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T$ denotes the closure of the range of D_T. To any contraction $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ one associates the unitary operator $U(T) : \mathcal{H}_1 \oplus \mathcal{D}_{T^*} \to \mathcal{H}_2 \oplus \mathcal{D}_T$ by the formula:

$$U(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}. \tag{2.4}$$

3. Lattice structures

Let Φ be a linear completely positive map on $\mathcal{M}_n$. The matrix $S = S_\phi$ given by (2.2) is positive, and by Theorem 1.5.3 in [2], there exists a uniquely determined family $\Gamma = \{\Gamma_{kj} \mid 1 \leq k \leq j \leq n^2\}$ of complex numbers with the following properties. Thus,

$$S_{kk} = \Gamma_{kk}, \quad 1 \leq k \leq n^2,$$

and for $1 \leq k < j \leq n^2$, $\Gamma_{kj} \in \mathcal{L}(\mathcal{D}_{\Gamma_{k+1,j}}, \mathcal{D}_{\Gamma^*_{k,j-1}})$ are contractions such that

$$S_{kj} = \Gamma_{kk}^{1/2}(R_{k,j-1}U_{k+1,j-1}C_{k+1,j} + D_{\Gamma^*_{k,k+1}} \dots D_{\Gamma^*_{k,j-1}}\Gamma_{kj}D_{\Gamma_{k+1,j}} \dots D_{\Gamma_{j-1,j}})\Gamma_{jj}^{1/2}. \tag{3.1}$$

We use the convention that $\mathcal{D}_{\Gamma_{kk}}$ is just (the closure of) the range of Γ_{kk}. This algorithm is valid in higher dimensions as well, that is the entries of S could be bounded operators and then the parameters Γ_{kj} would be also operators. The notation used in (3.1) is

quite involved but easy to explain. Also, this formula shows that each S_{kj} belongs to a certain disk.

For a fixed k, the operator R_{kj} which appears in (3.1) is the row contraction

$$R_{kj} = \left[\ \Gamma_{k,k+1}, \quad D_{\Gamma^*_{k,k+1}}\Gamma_{k,k+2}, \quad \dots\ , \quad D_{\Gamma^*_{k,k+1}} \dots D_{\Gamma^*_{k,j-1}}\Gamma_{kj}\ \right].$$

Analogously, for a fixed j, the operator C_{kj} is the column contraction

$$C_{kj} = \left[\ \Gamma_{j-1,j}, \quad \Gamma_{j-2,j}D_{\Gamma_{j-1,j}}, \quad \dots\ , \quad \Gamma_{kj}D_{\Gamma_{k-1,j}} \dots D_{\Gamma_{j-1,j}}\ \right]^t,$$

where "t" stands for matrix transpose. The operators U_{ij} are defined by the recursion: $U_{kk} = 1$ and for $j > k$,

$$U_{kj} = U_j(\Gamma_{j,j+1})U_j(\Gamma_{j,j+2}) \dots U_j(\Gamma_{kj})(U_{k+1,j} \oplus I_{\mathcal{D}_{\Gamma^*_{kj}}}),$$

where the subscript j at $U(\Gamma_{j,j+l})$ means that for $1 \leq l \leq j-k$ the unitary operator $U_j(\Gamma_{k,k+l})$ is defined from

$$\left(\oplus_{m=1}^{l-1}\mathcal{D}_{\Gamma_{k+1,k+m}}\right) \oplus (\mathcal{D}_{\Gamma_{k+1,k+l}} \oplus \mathcal{D}_{\Gamma^*_{k,k+l}}) \oplus \left(\oplus_{m=l+1}^{j}\mathcal{D}_{\Gamma_{k,k+m}}\right)$$

into

$$\left(\oplus_{m=1}^{l-1}\mathcal{D}_{\Gamma_{k+1,k+m}}\right) \oplus (\mathcal{D}_{\Gamma^*_{k,k+l-1}} \oplus \mathcal{D}_{\Gamma_{k,k+l}}) \oplus \left(\oplus_{m=l+1}^{j}\mathcal{D}_{\Gamma_{k,k+m}}\right)$$

by the formula

$$U_j(\Gamma_{k,k+l}) = I \oplus U(\Gamma_{k,k+l}) \oplus I.$$

We note that the above formula for U_{kj} comes from the familiar Euler factorization of $SO(N)$, [7].

We obtain the following result.

Theorem 3.1. *There exists a one-to-one correspondence between the set of linear completely positive maps on $\mathcal{M}_n$ and the families $\Gamma = \{\Gamma_{kj} \mid 1 \leq k \leq j \leq n^2\}$ of complex numbers such that $\Gamma_{kk} \geq 0$ for $1 \leq k \leq n^2$, and for $1 \leq k < j \leq n^2$, $\Gamma_{kj} \in \mathcal{L}(\mathcal{D}_{\Gamma_{k+1,j}}, \mathcal{D}_{\Gamma^*_{k,j-1}})$ are contractions. The correspondence is given by* (2.3) *and* (3.1)*.*

This result can be rephrased as a Schur-Cohn type test for complete positivity.

Algorithm 3.2. *Consider a linear map Φ on $\mathcal{M}_n$. The complete positivity of Φ can be verified as follows:*

(1) *Consider the matrix $S = S_\Phi$ given by formula* (2.2)*.*

(2) *Check $\Gamma_{kk} \geq 0$ for each k. If for some k, $\Gamma_{kk} < 0$, then Φ is not completely positive. If for some k, $\Gamma_{kk} = 0$, then the whole kth row (and column) of S must be zero.*

(3) *Calculate the numbers Γ_{kj} according to formula* (3.1)*. At each step check the condition $|\Gamma_{kj}| \leq 1$ and keep track of the compatibility condition $\Gamma_{kj} \in \mathcal{L}(\mathcal{D}_{\Gamma_{k+1,j}}, \mathcal{D}_{\Gamma^*_{k,j-1}})$. If this can be done for all indices kj, then Φ is completely positive. Otherwise, Φ is not completely positive.*

We illustrate the applicability of this algorithm for the case of completely positive maps on $\mathcal{M}_2$.

Example 3.3. A detailed analysis of quantum binary channels is given in [9]. We show here how Theorem 3.1 relates to that analysis. It is showed in [5] that any quantum binary channel Φ has a representation

$$\Phi(A) = U[\Phi_{\mathbf{t},\Lambda}(VAV^*)]U^*,$$

where $U, V \in U(2)$ and $\Phi_{\mathbf{t},\Lambda}$ has the matrix representation

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix}$$

with respect to the Pauli basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$ of $\mathcal{M}_2$. We can obtain (formula (26) in [9]) that

$$S_{\Phi_{\mathbf{t},\Lambda}} = \frac{1}{2}\begin{bmatrix} 1+t_3+\lambda_3 & t_1-it_2 & 0 & \lambda_1+\lambda_2 \\ t_1+it_2 & 1-t_3-\lambda_3 & \lambda_1-\lambda_2 & 0 \\ 0 & \lambda_1-\lambda_2 & 1+t_3-\lambda_3 & t_1-it_2 \\ \lambda_1+\lambda_2 & 0 & t_1+it_2 & 1-t_3+\lambda_3 \end{bmatrix}.$$

Similarly, by formula (27) in [9],

$$S_{\hat{\Phi}_{\mathbf{t},\Lambda}} = \frac{1}{2}\begin{bmatrix} 1+t_3+\lambda_3 & 0 & t_1+it_2 & \lambda_1+\lambda_2 \\ 0 & 1+t_3-\lambda_3 & \lambda_1-\lambda_2 & t_1+it_2 \\ t_1-it_2 & \lambda_1-\lambda_2 & 1-t_3-\lambda_3 & 0 \\ \lambda_1+\lambda_2 & t_1+it_2 & 0 & 1-t_3+\lambda_3 \end{bmatrix}.$$

It is slightly more convenient to deal with $S = [S_{kj}]_{k,j=1}^4 = 2S_{\hat{\Phi}_{\mathbf{t},\Lambda}}$. Formula (3.1) gives:

$$S_{11} = \Gamma_{11} = 1+t_3+\lambda_3; \qquad S_{22} = \Gamma_{22} = 1+t_3-\lambda_3;$$

$$S_{33} = \Gamma_{33} = 1-t_3-\lambda_3; \qquad S_{44} = \Gamma_{44} = 1-t_3+\lambda_3;$$

$$\Gamma_{12} = 0, \quad \Gamma_{34} = 0;$$

$$S_{23} = \Gamma_{22}^{1/2}\Gamma_{23}\Gamma_{33}^{1/2},$$

so that

$$\Gamma_{23} = \frac{\lambda_1-\lambda_2}{(1+t_3-\lambda_3)^{1/2}(1-t_3-\lambda_3)^{1/2}};$$

$$S_{13} = \Gamma_{11}^{1/2}\Gamma_{13}D_{\Gamma_{23}}\Gamma_{33}^{1/2},$$

so that

$$\Gamma_{13} = \frac{(t_1+it_2)(1+t_3-\lambda_3)^{1/2}}{((1+t_3-\lambda_3)(1-t_3-\lambda_3)-(\lambda_1-\lambda_2)^2)^{1/2}(1+t_3+\lambda_3)^{1/2}};$$

$$S_{24} = \Gamma_{22}^{1/2}D_{\Gamma_{23}^*}\Gamma_{24}\Gamma_{44}^{1/2},$$

so that

$$\Gamma_{24} = \frac{(t_1 + it_2)(1 - t_3 - \lambda_3)^{1/2}}{((1 + t_3 - \lambda_3)(1 - t_3 - \lambda_3) - (\lambda_1 - \lambda_2)^2)^{1/2}(1 - t_3 + \lambda_3)^{1/2}}.$$

Finally,

$$S_{14} = \Gamma_{11}^{1/2}(-\Gamma_{13}\Gamma_{23}^*\Gamma_{24} + D_{\Gamma_{13}^*}\Gamma_{14}D_{\Gamma_{24}})\Gamma_{44}^{1/2}.$$

By now, the formula for Γ_{14} becomes quite intricate, but there in no problem to write it explicitly. We deduce that $\Phi_{\mathbf{t},\Lambda}$ is completely positive if and only if the following eigth inequalities hold:

$$\Gamma_{kk} \geq 0, \quad k = 1, \dots, 4,$$
$$|\Gamma_{23}| \leq 1, \quad |\Gamma_{13}| \leq 1, \quad |\Gamma_{24}| \leq 1, \quad |\Gamma_{14}| \leq 1.$$

Also, we know what happens in the degenerate cases. Thus, the implication of $\Gamma_{kk} = 0$ for some k on the structure of $\Phi_{\mathbf{t},\Lambda}$ is clear. Also, if $|\Gamma_{23}| = 1$, then necessarily $t_1 = t_2 = 0$ and $\lambda_1 + \lambda_2 = (1 + t_3 + \lambda_3)^{1/2}\Gamma_{14}(1 - t_3 + \lambda_3)^{1/2}$ for some contraction Γ_{14}. If either $|\Gamma_{13}| = 1$ or $|\Gamma_{24}| = 1$, then necessarily $\Gamma_{14} = 0$ and $S_{14} = \Gamma_{11}^{1/2}(-\Gamma_{13}\Gamma_{23}^*\Gamma_{24})\Gamma_{44}^{1/2}$.

We notice that this result is of about the same nature as that in [9]. This is because the first step of (3.1) is precisely Lemma 6 in [9] which is used for the analysis in [9]. If we used the block version of (3.1) then we would deduce precisely Theorem 1 of [9]. What we basically have done here is that we used (3.1) in order to deduce in a systematic way the condition that $R_{\Phi_{\mathbf{t},\Lambda}}$ in Theorem 1 of [9] is a contraction. One advantage of doing this is that it works in higher dimensions.

We also have to note that the correspondence between S_Φ and the parameters Γ si nonlinear. Only for the first step the correspondence is affine and therefore can be used in the analysis of extreme points in the case $n = 2$, as it was done in [9]. This seems to be unclear for $n \geq 2$. $\square$

We conclude with the presentation of so-called lattice structures that can be associated to completely positive maps on $\mathcal{M}_n$. This comes from the remark that S_Φ has displacement structure as described in [10] and the general lattice structures associated to matrices with displacement structure in [10] can be used in our particular case. We can omit the details. In Figure 1 we show the lattice structure of completely positive maps on $\mathcal{M}_2$.

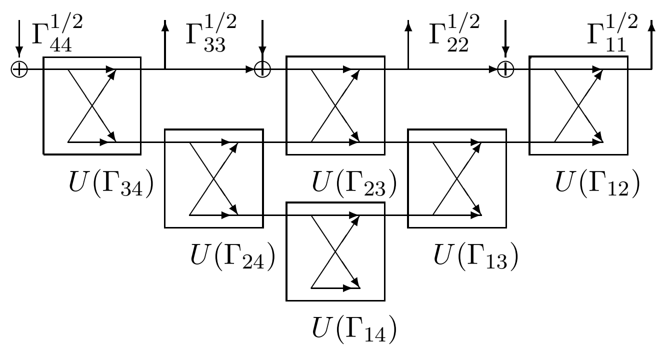

FIGURE 1. Lattice structure for completely positive maps on $\mathcal{M}_2$

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